Near-Linear Time Approximation Schemes for some Implicit Fractional Packing Problems

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Packing LP

Input: $v \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^m$ nonnegative

Objective:

$$\max \langle v, x \rangle \text{ over } x \in \mathbb{R}^n$$
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incidence matrix A dense

Packing LP

★ *Fast relative approximations*

Input: $v \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^m$ nonnegative

★ ***Error parameter:*** $\epsilon > 0$

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Objective: $\max \langle v, x \rangle$ over $x \in \mathbb{R}^n$

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s.t. $Ax \leq c$ and $x \geq \mathbb{0}$

★ ***Output:*** $x \geq \mathbb{0}^n$ s.t. $Ax \leq c$ and $\langle v, x \rangle \geq (1-\epsilon)\text{OPT}$

★ ***Run time:*** $O([\text{INPUT SIZE}]\,\text{poly}(1/\epsilon, \log[\text{INPUT}]))$

Related work (abbrev.)

- MWU / Lagrangian relax. faster than LP solvers
  Shahrokhi & Matula '90, Klein et al '94, Plotkin et al '95
  Grigoriadis & Khachiyan '94, Young '95, and many others

- Explicit pure packing and pure covering
  - $O(N + (m+n)\log(N)/\epsilon^2)$
    Koufogiannakis & Young '07
  - $\tilde{O}(N/\epsilon)$ *Allen-Zhu & Orrechia '15, Wang et al '16*

- Mixed packing and covering in $\tilde{O}(N/\epsilon^2)$
  Young '14

Tree packings and network strength

Input: graph $G = (V, E)$ w/ spanning trees $\mathcal{T}$
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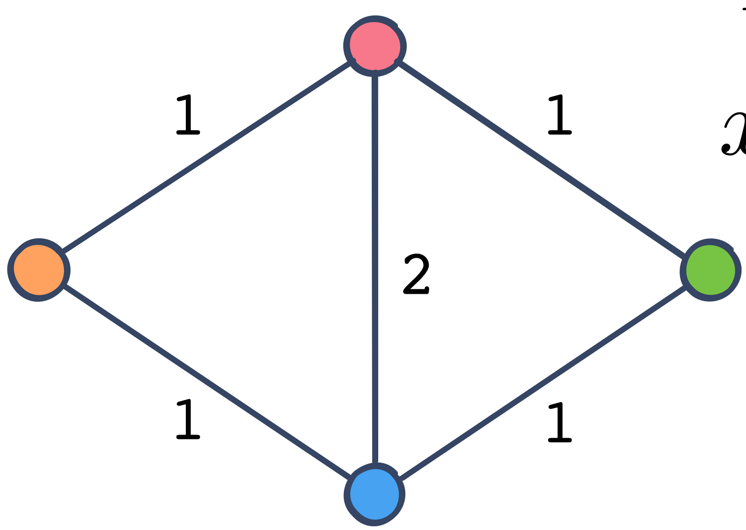

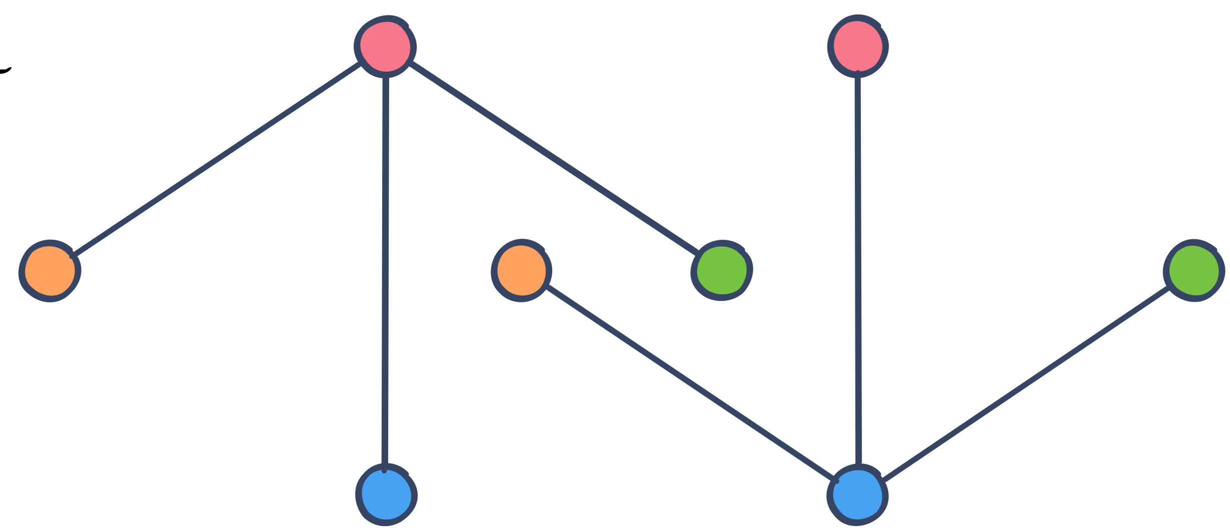

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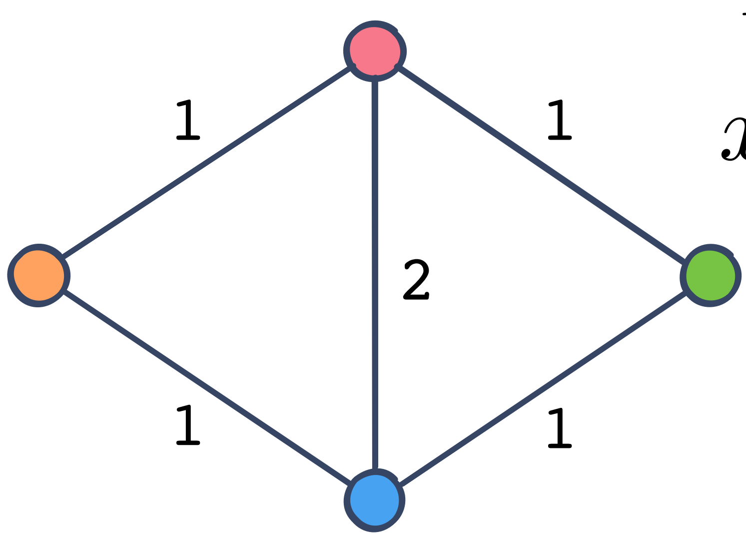

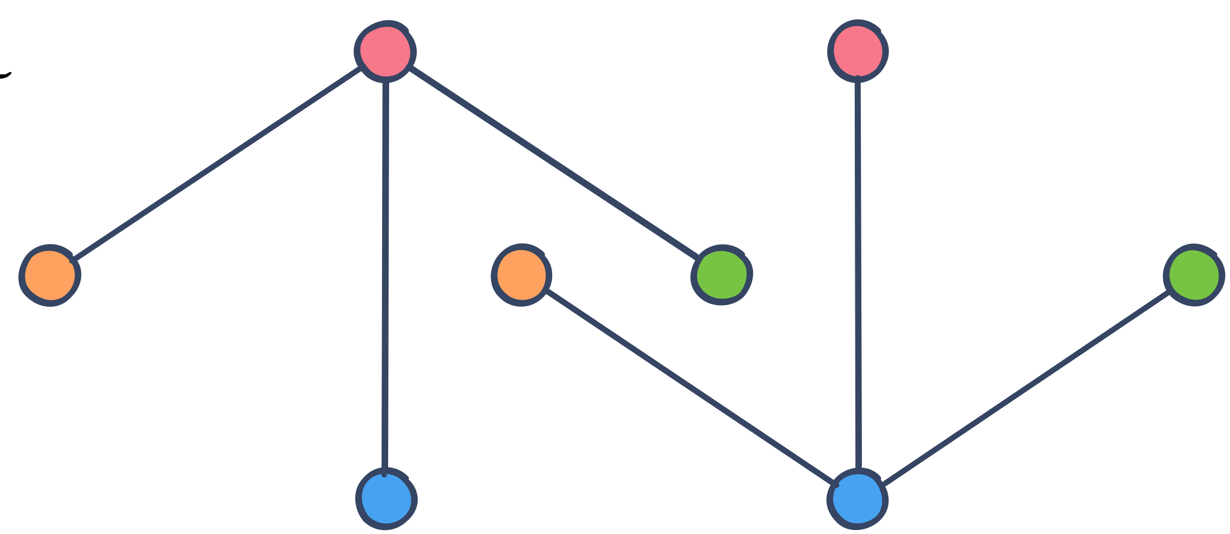

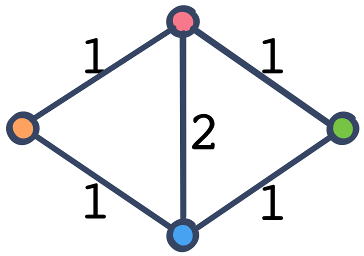

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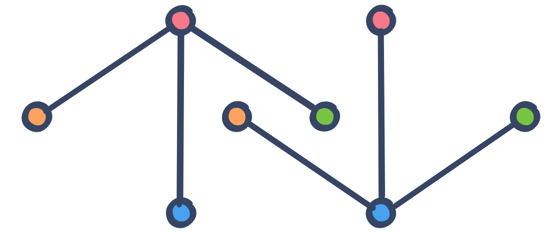

- Exact tree packings in $\tilde{O}(mn^3)$, $\tilde{O}(mn^2 \log C)$

  Gabow and Manu [1995,1998]

- $(1+\epsilon)$-apx tree packings in $\tilde{O}(m^2/\epsilon^2)$

  Garg and Khandekar

- Sparsification ➡ $(1+\epsilon)$-apx network strength in $O(m + n^{3/2}/\epsilon^4)$ time

  Karger [1993]

- Network strength ➡ 2-approx min-cut value

  Tutte, Nash-Williams [1961]

- $(2+\epsilon)$-apx min-cut value in $O(m/\epsilon)$ *Matula [1993]*

Scheduling and unsplittable flow

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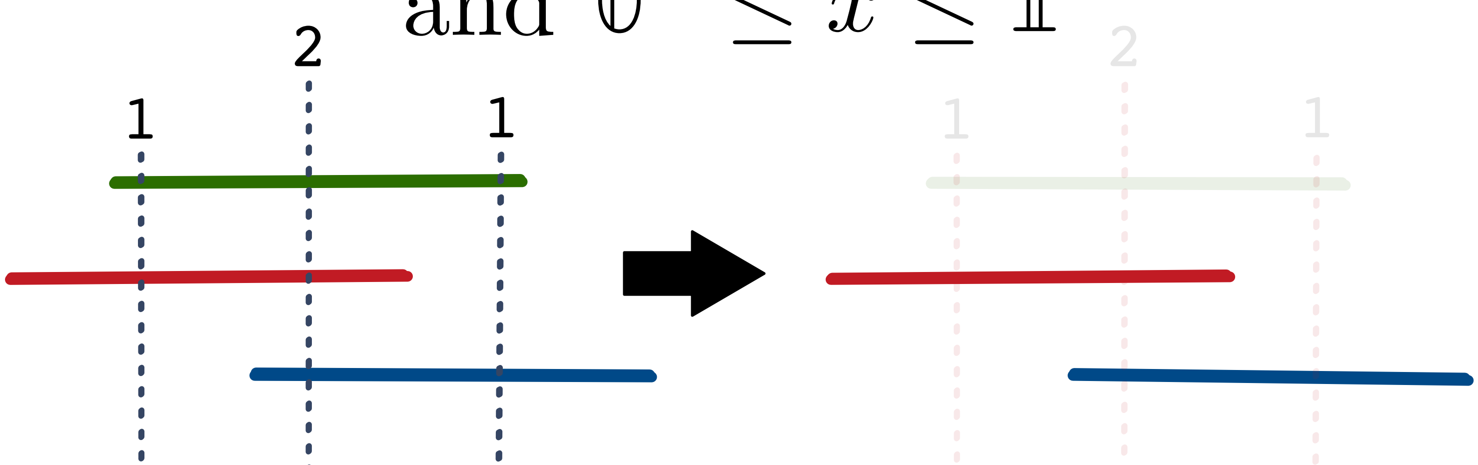

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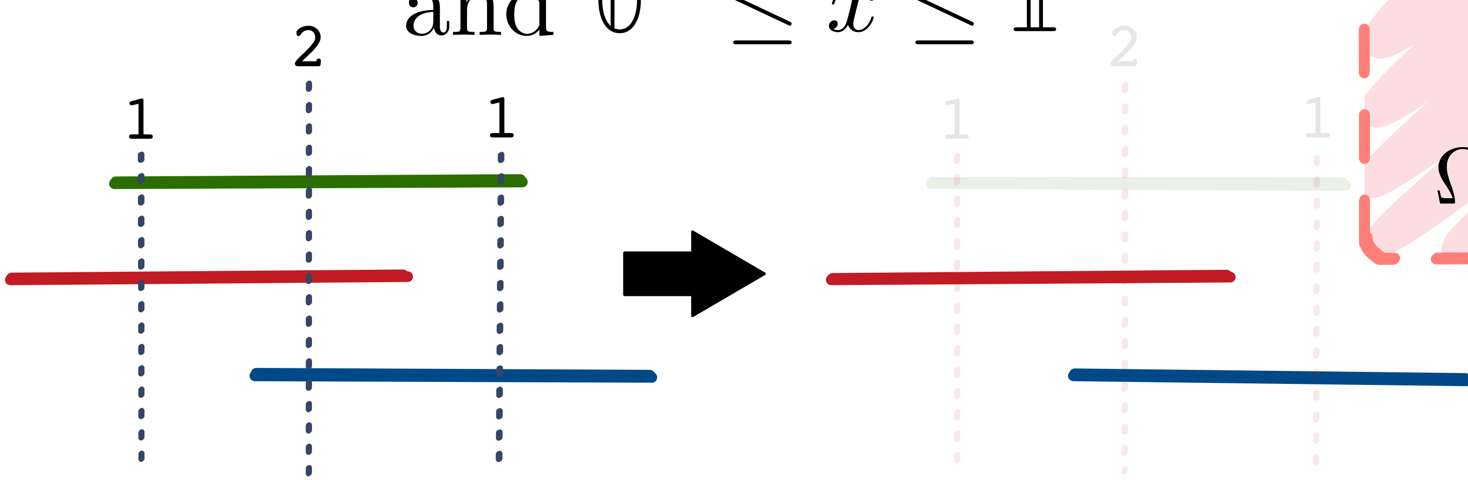
implicit incidence matrix A has $\Omega(mn)$ nonzeroes



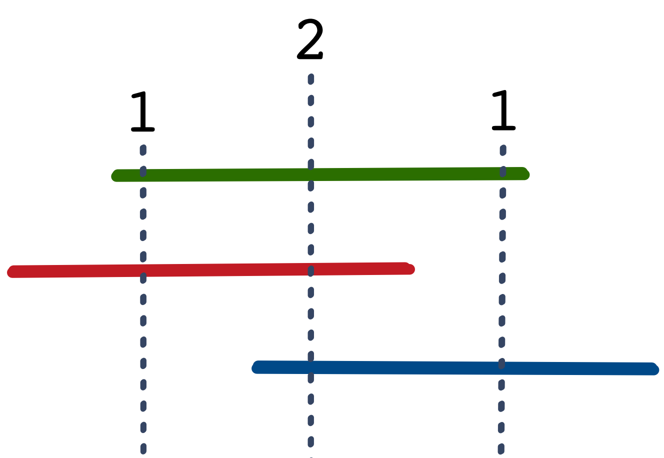
Related work (abbrev.)

- equivalent to unsplittable flow on paths

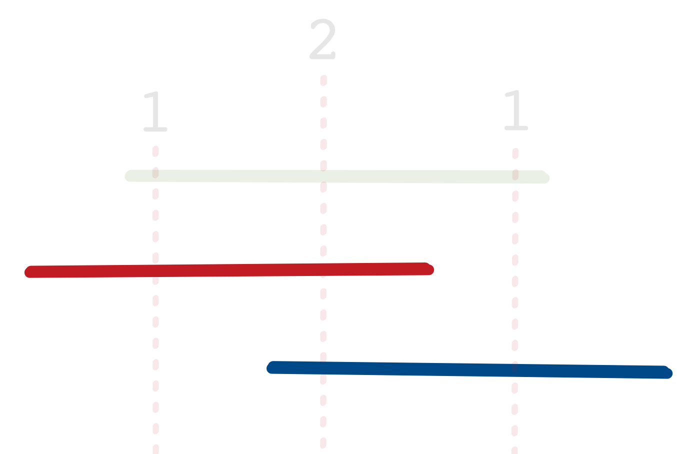
- demand $d = \mathbb{1}$ ➡ totally unimodular ➡ poly-time
  ⬇
  $O((m+n)^2 \log(m+n))$ via min-cost flow

  Arkin and Silverberg [1987]

- Open question by Borodin asks for FPTAS in nearly-linear time



➡



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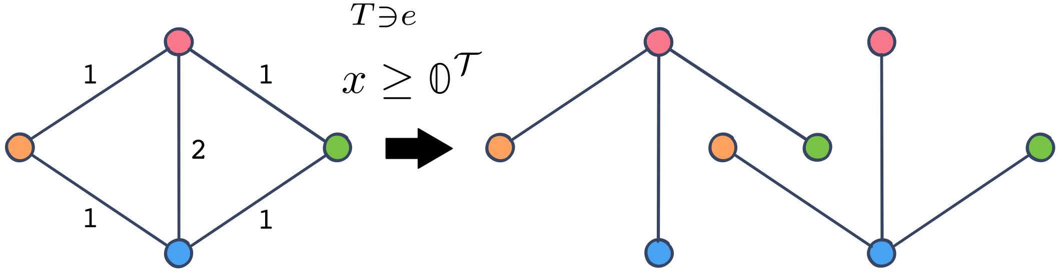

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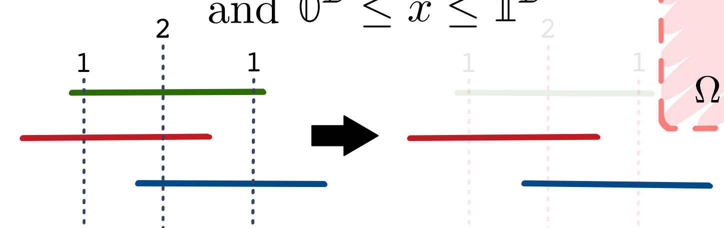

we obtain $(1 + \epsilon)$-relative approximations in time

$$\tilde{O}(m/\epsilon^2)$$

$$\tilde{O}((m+n)/\epsilon^2)$$

Basic design

multiplicative weight updates (MWU) + dynamic data structures

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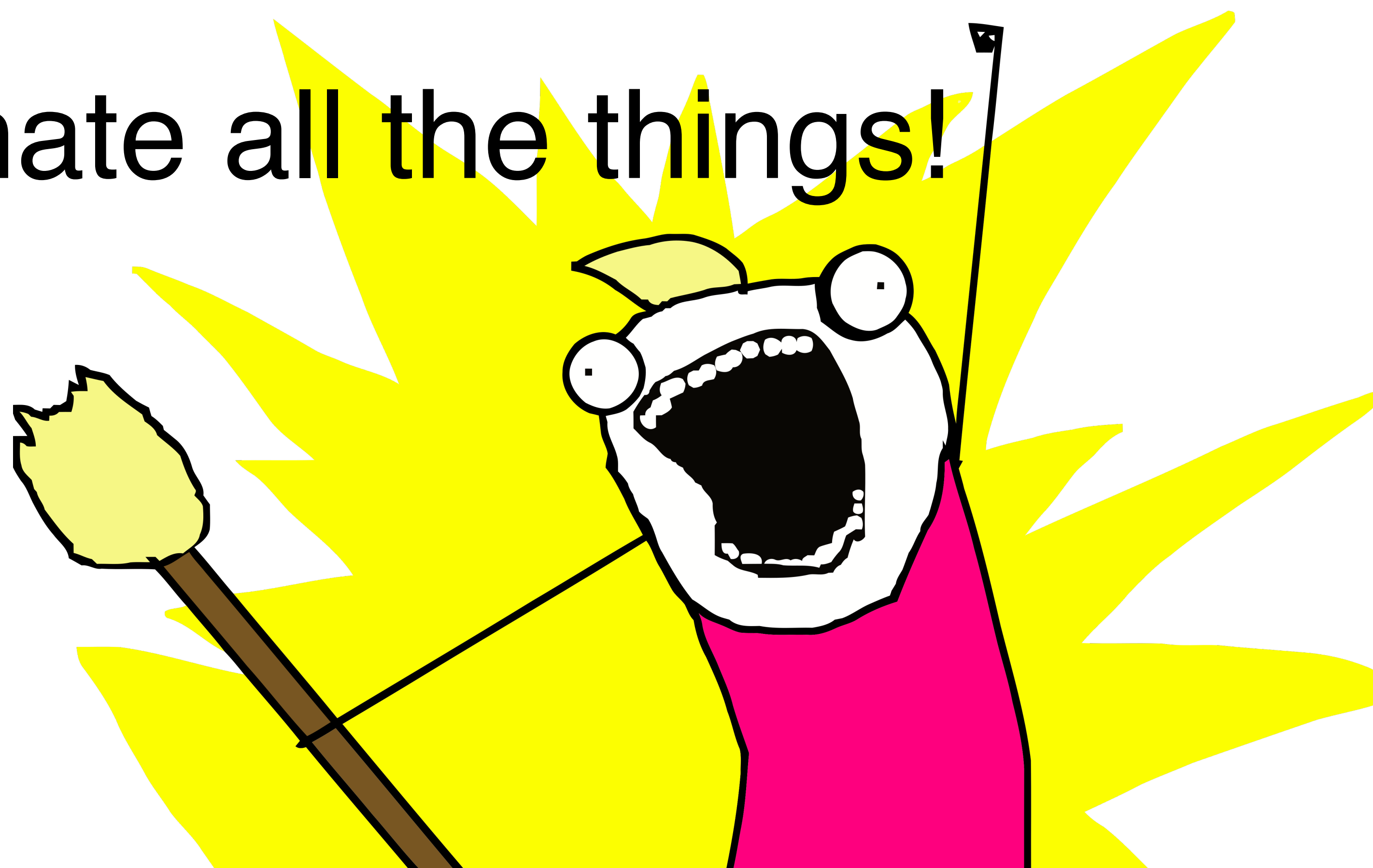
+ approximate all the things!



Multiplicative weight updates

packing problem → MWU → knapsack problems

$\max \langle v, x \rangle$ over $x \in \mathbb{R}^n$

s.t. $Ax \leq c$ and $x \geq \mathbb{0}$

0 initialize weights $w \leftarrow 1/c$

1 solve knapsack

$\max \langle v, x \rangle$

s.t. $\langle w, Ax \rangle \leq \langle w, c \rangle$

and $x \geq \mathbb{0}$

$\tilde{O}(m/\epsilon^2)$ iterations

2 update weights w

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simple greedy solution

1a $i \leftarrow \arg\max_{i \in n} \frac{v_i}{\langle w, Ae_i\rangle}$

1b $x \leftarrow \frac{\langle w, c\rangle}{\langle w, Ae_i\rangle} e_i$

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MWU

minimum spanning trees

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1a $T \leftarrow \mathrm{MST}(w)$

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1a $T \leftarrow \mathrm{MST}(w)$

update T dynamically in $\tilde{O}(1)$ per edge update

[Holm et al 1998]

0 `initialize edge weights` $w \leftarrow 1/c$

1 **a** $T \leftarrow \mathrm{MST}(w)$

b $x \leftarrow \frac{\langle w, c \rangle}{\sum_{e \in T} w_e} e_T$

$\tilde{O}(m/\epsilon^2)$ `iterations`

2 `increase weight` w_e `for each` $e \in T$

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$\tilde{O}(m/\epsilon^2)$ iterations

$\times \quad n$ weights to update per iteration

$\tilde{O}(mn/\epsilon^2)$ time spent updating weights

(too slow!)

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naive: $\tilde{O}(mn/\epsilon^2)$ updates

A suffices to $(1+\epsilon)$-rel. approximate weights

B lazy amort. updates

➡ $\tilde{O}(m/\epsilon^2)$ total updates

[Young 2014]

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total running time

$\tilde{O}(1)$ per weight update

$\times\ \tilde{O}(m/\epsilon^2)$ total updates

★ $\tilde{O}(m/\epsilon^2)$ running time

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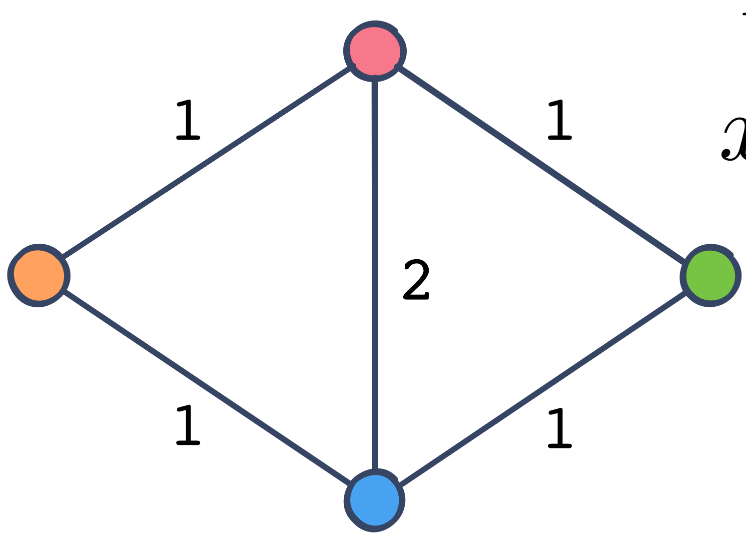

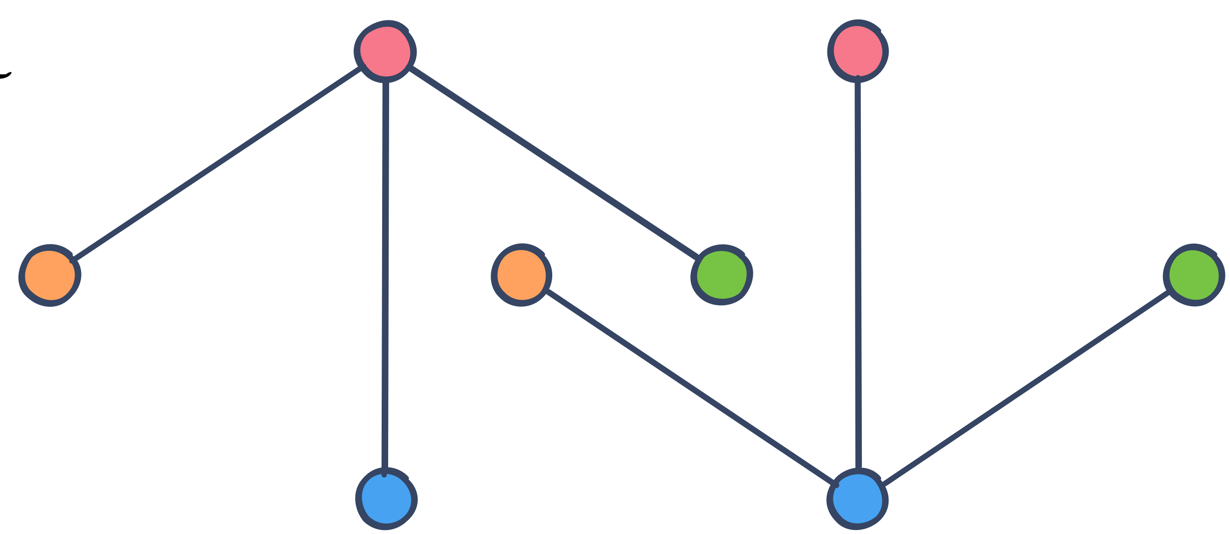

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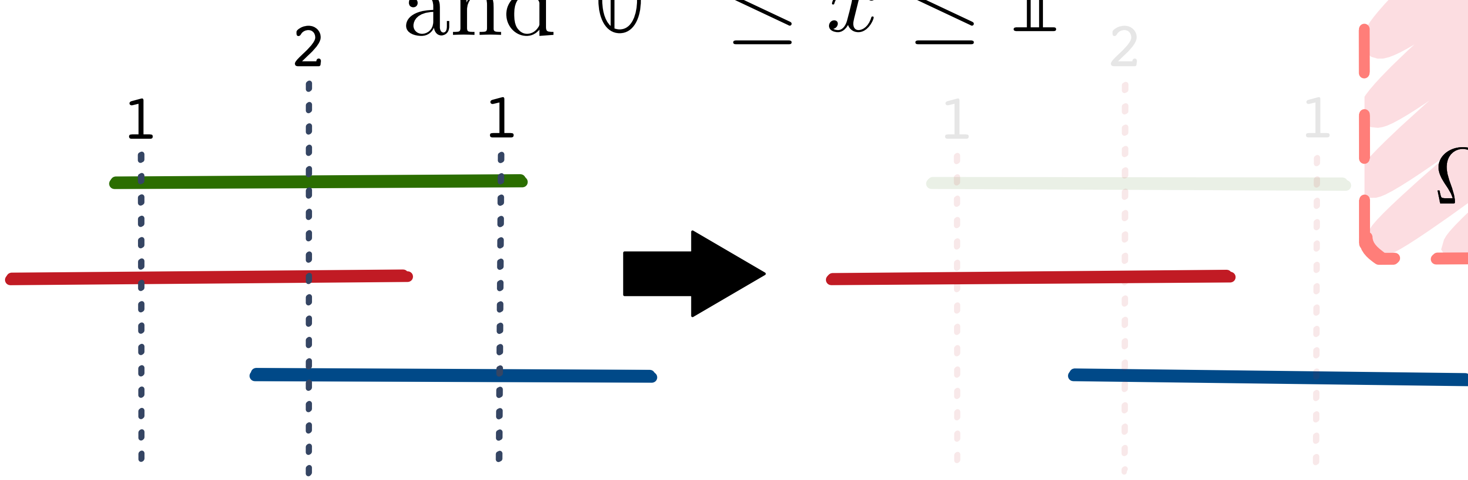

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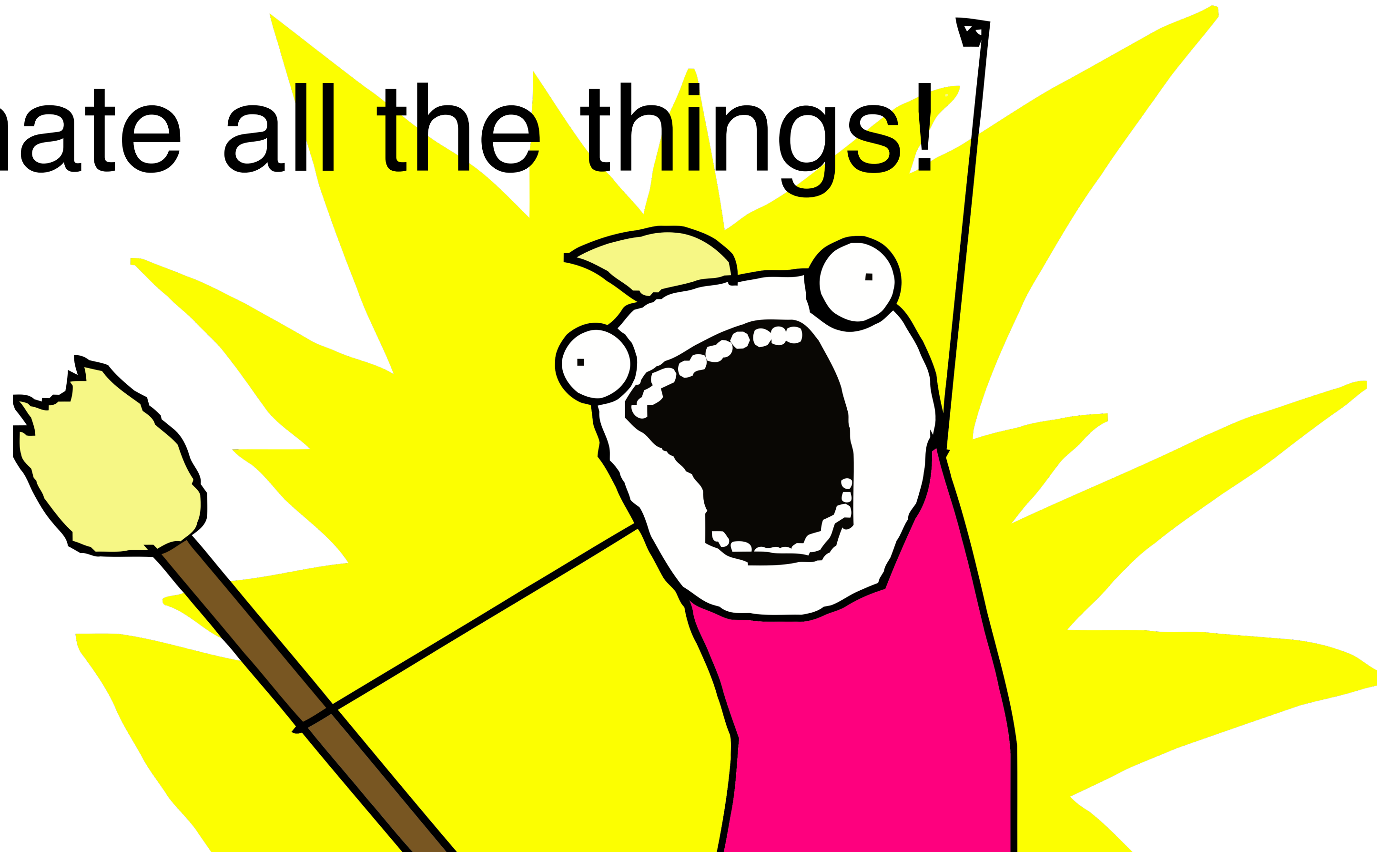

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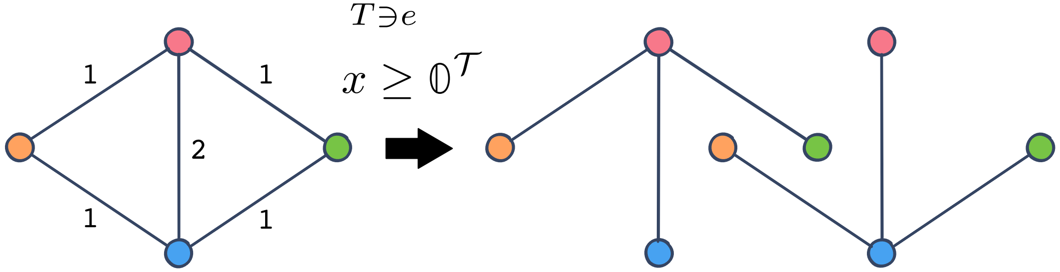

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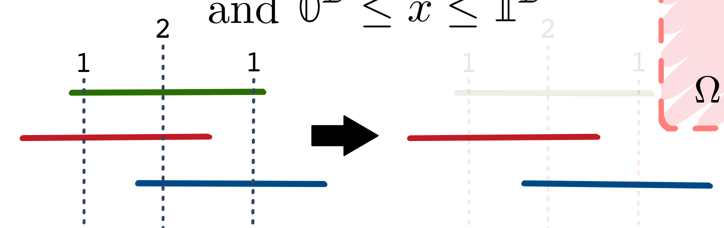

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